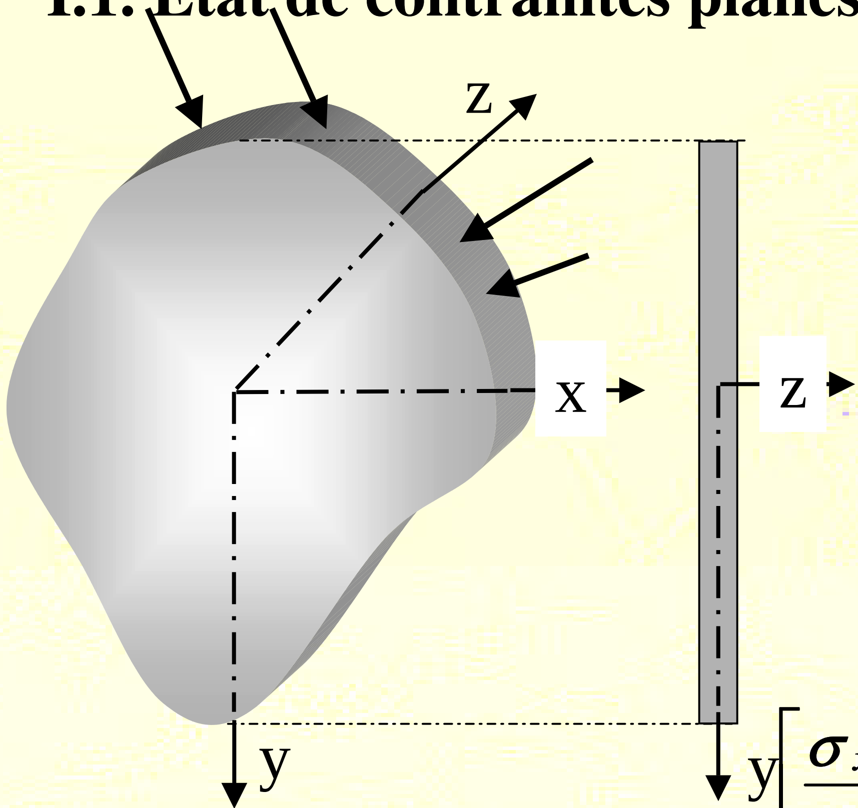
M

CHAPITRE VII
élasticité plane

I. Les deux états de l'élasticité plane

I.1. Etat de contraintes planes

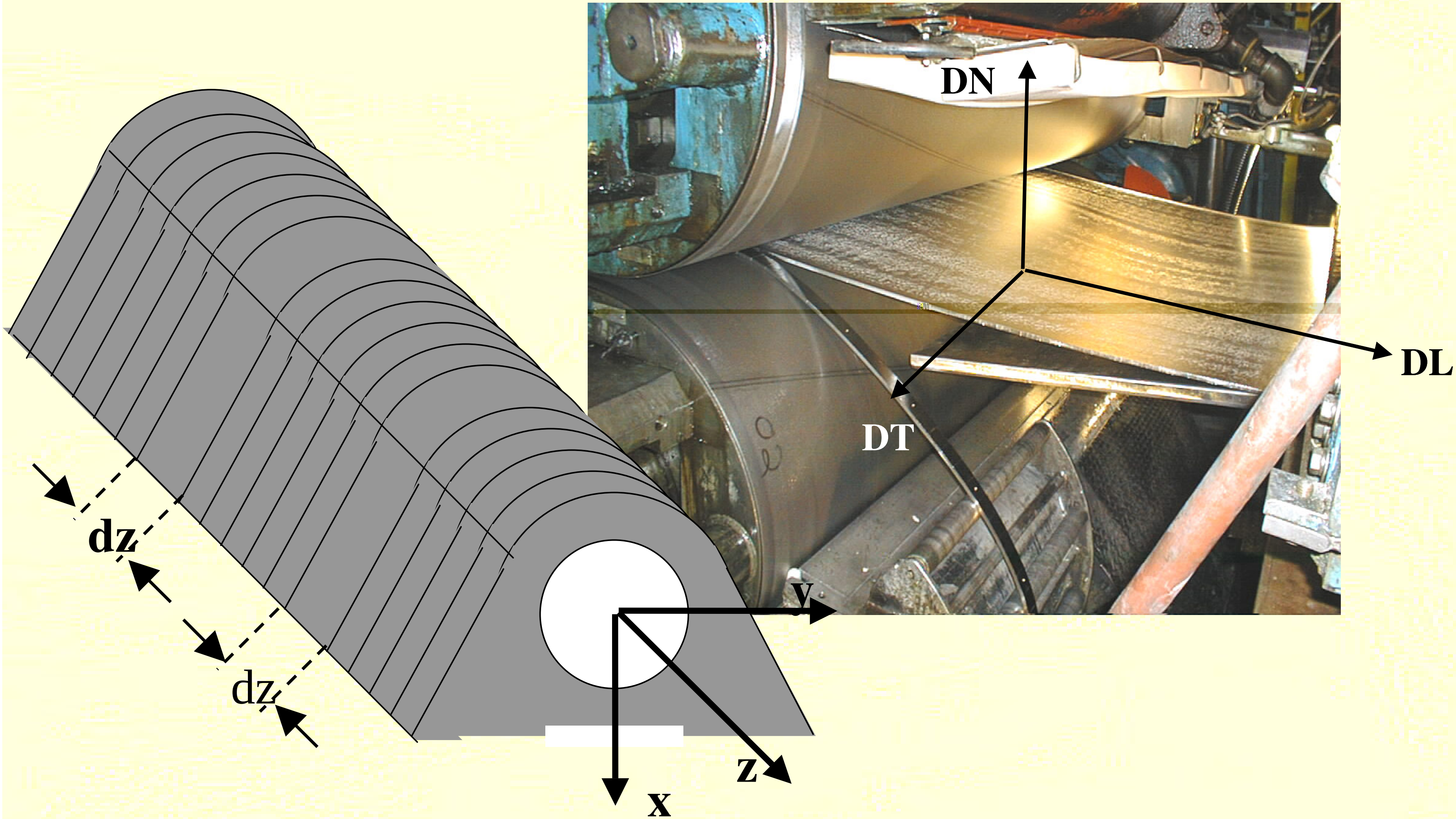


$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \dfrac{\sigma_{xx} - \nu\sigma_{yy}}{E} & \dfrac{\sigma_{xy}}{2G} & 0 \\ \dfrac{\sigma_{xy}}{2G} & \dfrac{\sigma_{yy} - \nu\sigma_{xx}}{E} & 0 \\ 0 & 0 & -\dfrac{\nu(\sigma_{xx} + \sigma_{yy})}{E} \end{bmatrix}$$

I. Les deux états de l'élasticité plane

I.2. Etat de déformations planes



I. Les deux états de l'élasticité plane

I.2. Etat de déformations planes

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] = 0 \Rightarrow \sigma_{zz} = \nu\left(\sigma_{xx} + \sigma_{yy}\right)$$

$$\begin{cases} \sigma_{xx} = \dfrac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy}\right] \\ \sigma_{xx} = \dfrac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{xx} + (1-\nu)\varepsilon_{yy}\right] \\ \sigma_{zz} = \nu\left(\sigma_{xx} + \sigma_{yy}\right) \end{cases}$$

I. Les deux états de l'élasticité plane

I.2. Etat de déformations planes

Contraintes planes

↓

déformationsplanes

$$E \rightarrow \frac{E}{\left(1-\upsilon^2\right)} \quad \upsilon \rightarrow \frac{\upsilon}{\left(1-\upsilon\right)}$$

Déformations planes

↓

Contraintes planes

$$E \rightarrow \frac{E\left(1+2\upsilon\right)}{\left(1+\upsilon\right)^2} \quad \upsilon \rightarrow \frac{\upsilon}{\left(1+\upsilon\right)}$$

II. Les équations fondamentales de l'élasticité plane

II.1. Equations de compatibilité

$$\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\varepsilon_{ij}+\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\varepsilon_{kk}-\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\varepsilon_{ik}-\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}\varepsilon_{jk}=0$$

$$\Downarrow$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2}+\frac{\partial^2 \varepsilon_{yy}}{\partial x^2}-2\frac{\partial^2 \varepsilon_{xy}}{\partial x\ \partial y}=0$$

III. La fonction d'Airy

III.1. Forces de volume nulles

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} = 0 \\[2ex] \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} = 0 \end{cases}$$

$$\begin{cases} \sigma_{xx} = \dfrac{\partial^2 \Phi}{\partial y^2} \\[2ex] \sigma_{yy} = \dfrac{\partial^2 \Phi}{\partial x^2} \\[2ex] \sigma_{xy} = -\dfrac{\partial^2 \Phi}{\partial x \partial y} \end{cases}$$

III. La fonction d'Airy

III.1. Forces de volume nulles

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right)=\frac{\partial^4\Phi}{\partial x^4}+2\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{\partial^4\Phi}{\partial y^4}=0$$

III. La fonction d'Airy

III.2. Forces dérivant d'un potentiel

$$\vec{F} = \overrightarrow{grad}V = \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \dfrac{\partial V}{\partial x} \\ \dfrac{\partial V}{\partial y} \end{bmatrix}$$

σ

$$\begin{cases} \dfrac{\partial\left(\;_{xx}+V\right)}{\partial x} \quad \dfrac{\partial\;_{xy}}{\partial y} \\ \dfrac{\partial\;_{xy}}{\partial x} + \dfrac{\partial\left(\;_{yy}+V\right)}{\partial y} = 0 \end{cases}$$

III. La fonction d'Airy

III.2. Forces dérivant d'un potentiel

$$\begin{cases} \dfrac{\partial\left(\sigma_{xx}+V\right)}{\partial x}+\dfrac{\partial \sigma_{xy}}{\partial y}=0 \\ \dfrac{\partial \sigma_{xy}}{\partial x}+\dfrac{\partial\left(\sigma_{yy}+V\right)}{\partial y}=0 \end{cases}$$

III. La fonction d'Airy

III.2. Forces dérivant d'un potentiel

$$\begin{cases} \sigma_{xx} + V = \dfrac{\partial^2 \Phi}{\partial y^2} \\ \sigma_{yy} + V = \dfrac{\partial^2 \Phi}{\partial x^2} \\ \sigma_{xy} = -\dfrac{\partial^2 \Phi}{\partial x \partial y} \end{cases}$$

Déformations planes

$$\Delta\Delta\Phi = \frac{1-2\upsilon}{1-\upsilon}\Delta V$$

Contraintes planes

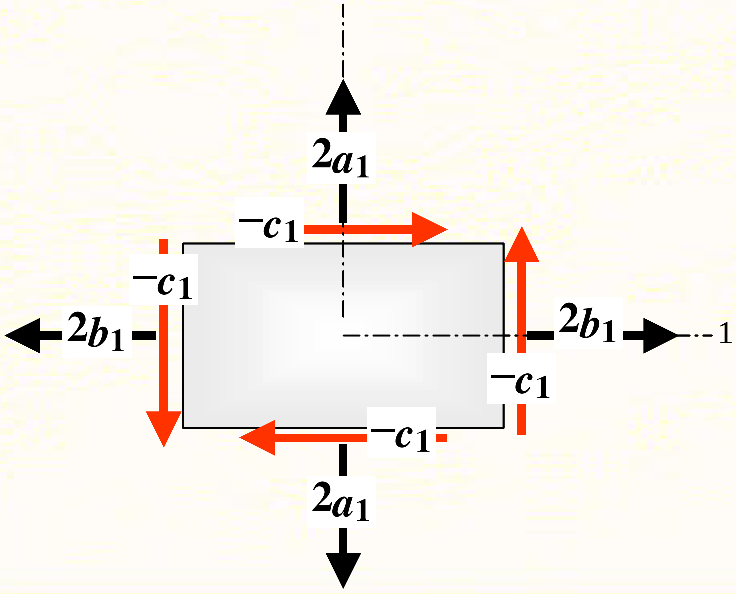
$$\Delta\Delta\Phi = (1-\upsilon)\Delta V$$

IV. Application à la théorie des poutres

IV.1. Solution de l'équation biharmonique par des polynômes

$$\frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0$$

Polynôme biquadratique $\varphi_2(x,y) = a_2 x^2 + b_2 xy + c_2 y^2$



$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix} = \begin{pmatrix} 2c_2 & -b_2 \\ -b_2 & 2a_2 \end{pmatrix}$$

IV. Application à la théorie des poutres

IV.1. Solution de l'équation biharmonique par des polynômes

$$\frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0$$

Polynôme bicubique $\varphi_3(x, y) = a_3 x^3 + b_3 x^2 y + c_3 x y^2 + d_3 y^3$

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix} = \begin{pmatrix} 2c_3 x + 6d_3 y & -2b_3 x - 2c_3 y \\ -2b_3 x - 2c_3 y & 6a_3 x + 2b_3 y \end{pmatrix}$$

IV. Application à la théorie des poutres

IV.2. Détermination des déplacements

Contraintes planes

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \dfrac{\sigma_{xx} - \nu\sigma_{yy}}{E} & \dfrac{\sigma_{xy}}{2G} & 0 \\ \dfrac{\sigma_{xy}}{2G} & \dfrac{\sigma_{yy} - \nu\sigma_{xx}}{E} & 0 \\ 0 & 0 & -\dfrac{\nu\left(\sigma_{xx} + \sigma_{yy}\right)}{E} \end{bmatrix}$$

Déformations planes

$$\begin{cases} \varepsilon_{xx} = \dfrac{(1+\nu)}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right] \\ \varepsilon_{yy} = \dfrac{(1+\nu)}{E}\left[-\nu\sigma_{xx} + (1-\nu)\sigma_{yy}\right] \end{cases}$$

IV. Application à la théorie des poutres

IV.2. Détermination des déplacements

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{\partial u(x,y)}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial v(x,y)}{\partial y} \end{aligned}\right\}$$

$$u(x,y) \qquad {}_{xx}dx \quad f_1(y)$$

$$v(x,y) \qquad {}_{yy}dy \quad f_2(x)$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u(x,y)}{\partial y} + \frac{\partial v(x,y)}{\partial x}\right)$$

IV. Application à la théorie des poutres

IV.2. Détermination des déplacements

$$\begin{cases} u(x,y) = \int \varepsilon_{xx} dx + f_1(y) \\ v(x,y) = \int \varepsilon_{yy} dy + f_2(x) \end{cases} \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u(x,y)}{\partial y} + \frac{\partial v(x,y)}{\partial x} \right)$$

$$\Downarrow$$

$$u_1 = a + by \qquad v_1 = c - bx$$

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{xy} = 0$$