# Mécanique des milieux continus

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# Chapitre V : Méthodes énergétiques

**Chercher la meilleure solution aux équations de l'élasticité**

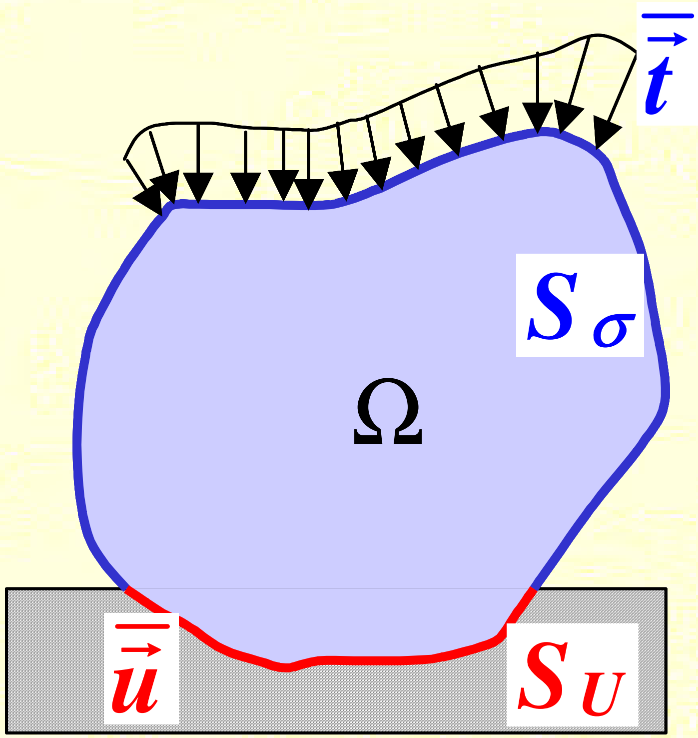
$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{L}}}} \; \underline{\underline{\varepsilon}}$$

$$\underbrace{\begin{cases} div\left(\underline{\underline{\sigma}}\right) + \vec{f} = \quad \underline{\underline{\sigma}}^{T} = \underline{\underline{\sigma}} \; \forall \vec{x} \in \Omega \\ \underline{\underline{\sigma}}\vec{n} = \vec{t}^{\,imposé} = \vec{\bar{t}} \quad \forall \vec{x} \in S_{\sigma} \end{cases}}_{\text{satisfait approximativement}}$$

**ou**

$$\underbrace{\begin{cases} \underline{\underline{\varepsilon}} = \frac{}{2}\left[\nabla\vec{u} + \left(\nabla\vec{u}\right)^{T}\right] \forall \vec{x} \in \Omega \\ \vec{u} = \vec{u}^{\,imposé} = \vec{\bar{u}} \quad \forall \vec{x} \in S_{U} \end{cases}}_{\textit{satisfait}\ \text{approximativement}}$$

# I. Champ de déplacement cinématiquement admissible et champ de contrainte statiquement admissible




$$\begin{cases} \underline{\underline{\varepsilon}}^{CA} = \frac{1}{2}\left[\nabla \vec{u}^{CA} + \left(\nabla \vec{u}^{CA}\right)^T\right] \forall \vec{x} \in \Omega \\ \vec{u}^{CA} = \vec{u}^{imposé} = \overline{\vec{u}} \quad \forall \vec{x} \in S_U \end{cases}$$

$$\begin{cases} div\left(\underline{\underline{\sigma}}^{SA}\right) + \vec{f} = \quad \underline{\underline{\sigma}}^{SA^T} = \underline{\underline{\sigma}}^{SA} \quad \forall \vec{x} \in \Omega \\ \underline{\underline{\sigma}}^{SA} \vec{n} = \vec{t}^{imposé} = \overline{\vec{t}} \quad \forall \vec{x} \in S_\sigma \end{cases}$$

**II. Equation des travau virtuels**

**II. . Conservation de l'énergie**

**énergie stockée dans le atériau**

$$\overbrace{\int_{\Omega} \underline{\underline{\sigma}}^{SA} \; \underline{\underline{\varepsilon}}^{CA} dv = \int_{\Omega} \left(\vec{\sigma}^{SA}\right)^{T} \vec{\varepsilon}^{CA} \, dv}$$

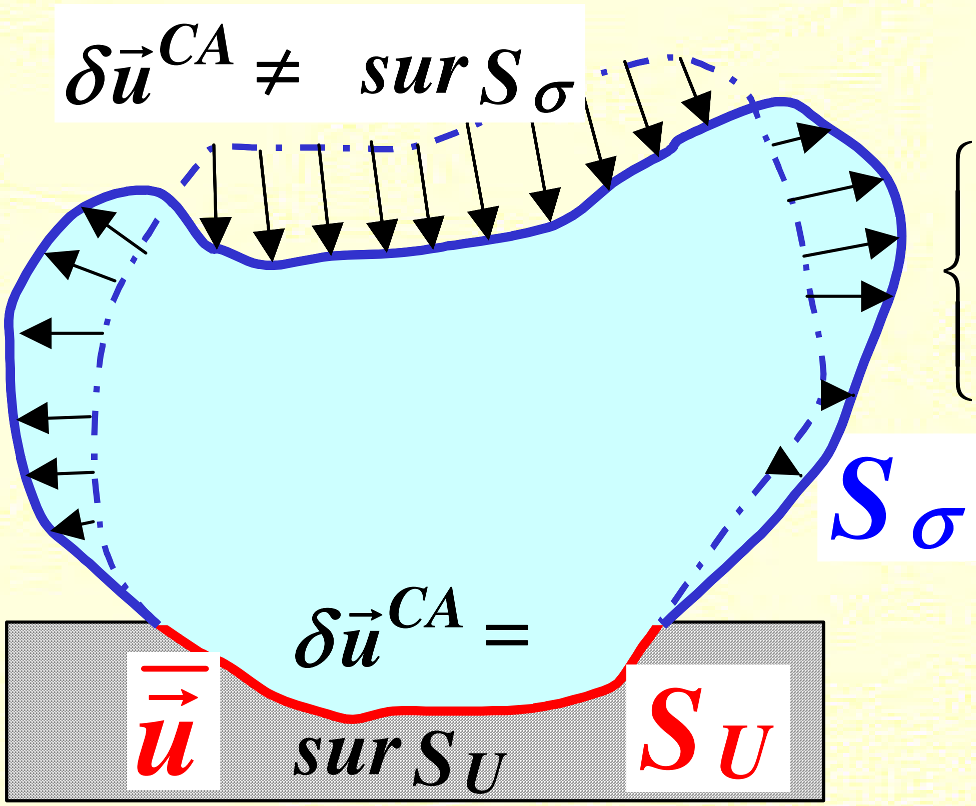
$$\underbrace{= \int_{S_\sigma} \bar{\vec{t}}^{T} \vec{u}^{CA} d S_\sigma + \int_{S_U} \vec{t}^{T} \bar{\vec{u}} \, d S_U + \int_{\Omega} \vec{f}^{T} \vec{u}^{CA} d\Omega} \qquad \left\{ \begin{array}{l} \forall \vec{u}^{CA} \\ \forall \vec{\sigma}^{SA} \end{array} \right.$$

**énergie fournie au atériau**

$$\underline{\underline{\sigma}} \neq \underline{\underline{\underline{\underline{L}}}} \; \underline{\underline{\varepsilon}}$$

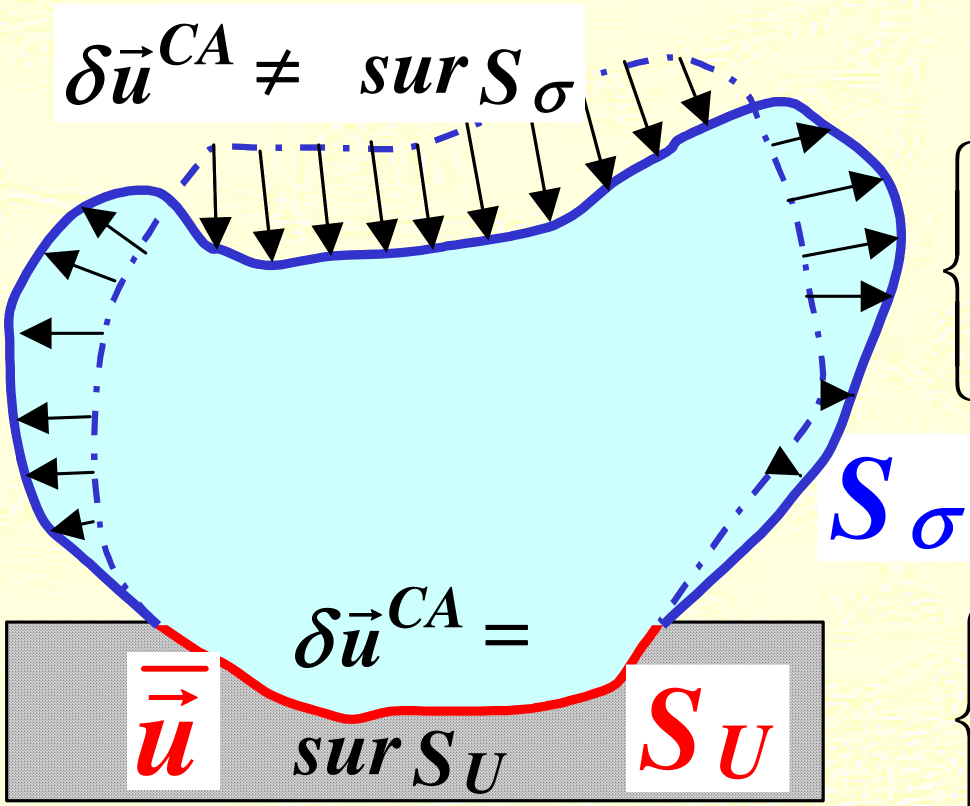
**II. Equation des travau virtuels**

**II2. Principe des travau virtuels des déplace ents**




$$\begin{cases} \underline{\underline{\varepsilon}}^{CA} = \frac{}{2}\left[\nabla\vec{u}^{CA} + \left(\nabla\vec{u}^{CA}\right)^T\right] \forall\vec{x} \in \Omega \\ \vec{u}^{CA} = \vec{u}^{imposé} = \overline{\vec{u}} \quad \forall\vec{x} \in S_U \end{cases}$$

## II. Equation des travau virtuels
## II2 . Principe des travau virtuels des déplace ents




$$\begin{cases} \underline{\underline{\varepsilon}}^{CA} = \frac{\;}{2}\left[\nabla\vec{u}^{CA} + \left(\nabla\vec{u}^{CA}\right)^T\right] \forall\vec{x}\in\Omega \\ \vec{u}^{CA} = \vec{u}^{imposé} = \overline{\vec{u}} \quad \forall\vec{x}\in S_U \end{cases}$$

$$\underbrace{\begin{cases} div\left(\underline{\underline{\sigma}}\right) + \vec{f} = \quad \underline{\underline{\sigma}}^T = \underline{\underline{\sigma}}\ \forall\vec{x}\in\Omega \\ \underline{\underline{\sigma}}\vec{n} \ = \vec{t}^{imposé} = \overline{\vec{t}} \quad \forall\vec{x}\in S_\sigma \end{cases}}_{\text{satisfait appro i ative ent}}$$

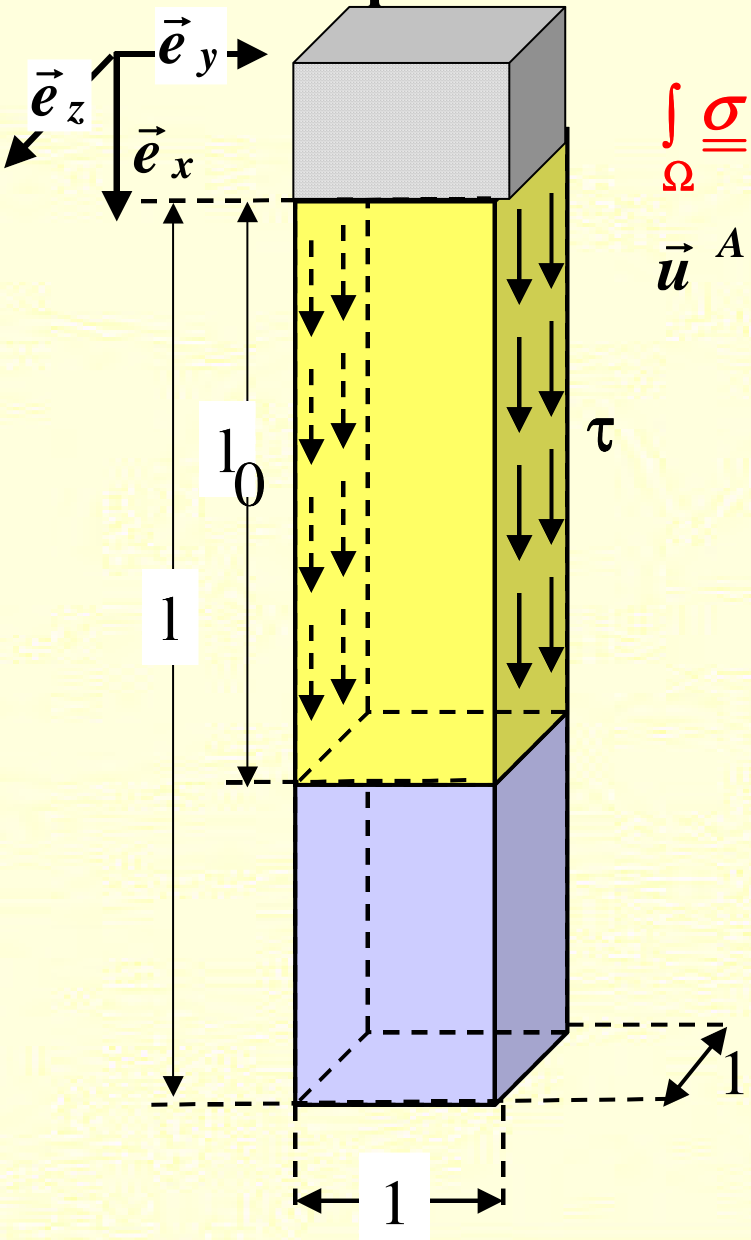
. Equation d

Principe d

en équilibre

## II. Equation des travau virtuels

### II2. Principe des travau virtuels des déplace ents e e ple




$$\int_{\Omega} \underline{\underline{\sigma}} \quad \delta\underline{\underline{\varepsilon}}^{CA} dv = \int_{S_\sigma} \vec{t}^{\,T} \delta\vec{u}^{CA} d S_\sigma + \int_{\Omega} \vec{f}^{\,T} \delta\vec{u}^{CA} d\Omega$$

$$\vec{u}^{\,A} = ax\vec{e}_x \quad \delta\vec{u}^{\,A} = \delta a x\vec{e}_x \quad \varepsilon_{xx} = a$$

$$\sigma_{xx} = E\varepsilon_{xx} = Ea$$

$$\int_{\Omega} \underline{\underline{\sigma}} \quad \delta\underline{\underline{\varepsilon}}^{CA} dv = \int_{\Omega} Ea(\delta a) dv$$

$$= Ea\Omega(\delta a) = Ea(\delta a) l$$

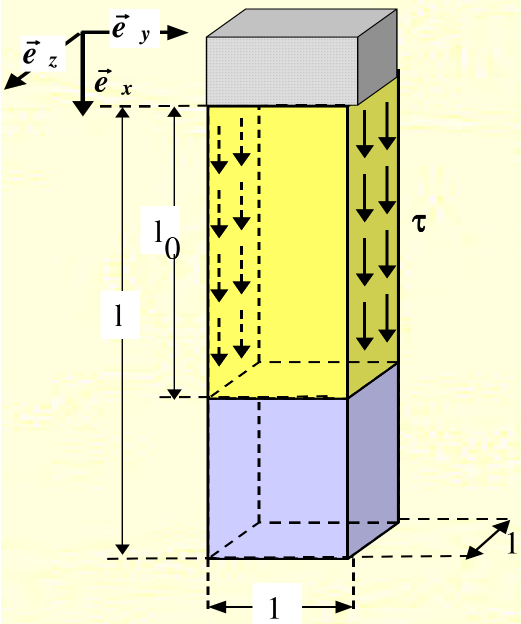
$$\int_{S_\sigma} \vec{t}^{\,T} \delta\vec{u}^{CA} d S_\sigma = \int_{S_\sigma} \tau(\delta a) x d S_\sigma$$

$$= \tau(\delta a) \int_{S_\sigma} x d S_\sigma = \frac{\tau(\delta a) l^2}{2}$$

## II. Equation des travau virtuels

## II2. Principe des travau virtuels des déplace ents e e ple

$$\int_{\Omega} \underline{\underline{\sigma}} \quad \delta \underline{\underline{\varepsilon}}^{CA} dv = \int_{S_\sigma} \vec{t}^{\,T} \delta \vec{u}^{CA} d S_\sigma + \int_{\Omega} \vec{f}^{\,T} \delta \vec{u}^{CA} d\Omega$$

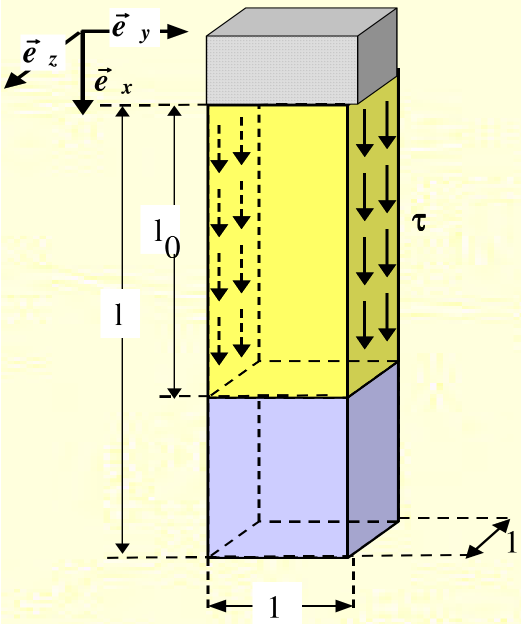



$$\left.\begin{aligned} &\int_{\Omega} \underline{\underline{\sigma}} \quad \delta \underline{\underline{\varepsilon}}^{CA} dv = Ea(\delta a) l \\ &= \int_{S_\sigma} \vec{t}^{\,T} \delta \vec{u}^{CA} d S_\sigma = \frac{\tau(\delta a) l^2}{2} \end{aligned}\right\} \Rightarrow a = \tau \frac{l}{2E}$$

$$\vec{u}^{\,A} = \frac{\tau l \ x}{2E} \vec{e}_x \qquad \sigma_{xx} = \frac{l \ \tau}{2}$$

## II. Equation des travau virtuels

## II2. Principe des travau virtuels des déplace ents e e ple

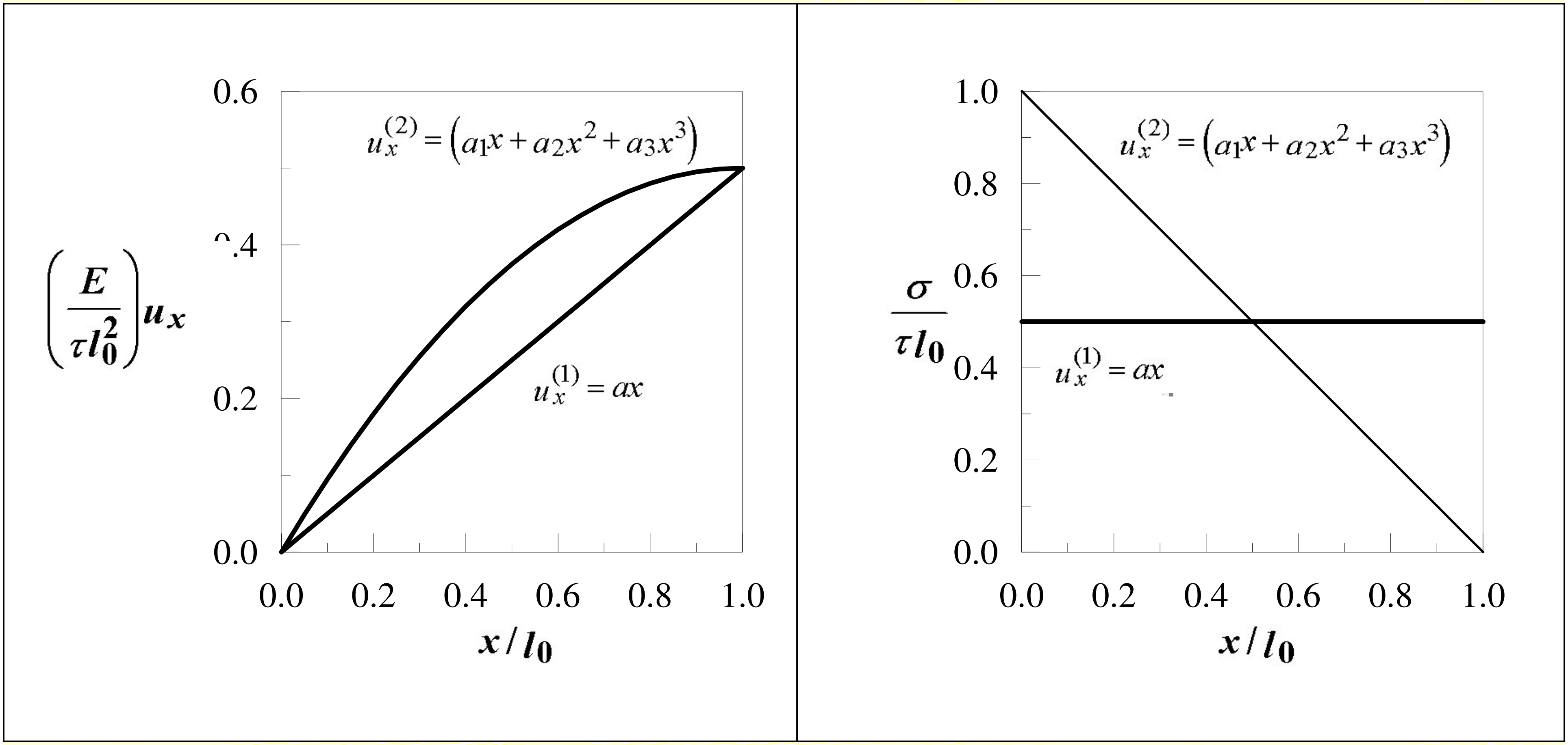



Est-ce la bonne solution ?

$$\vec u^{A} = \frac{\tau l \; x}{2E}\vec e_x \quad \sigma_{xx} = \frac{l \; \tau}{2}$$

$$\vec u^{B} = \left(a \; x + a_2 x^2 + a_3 x^3\right)\vec e_x$$

*est* eilleure ?



0.6
0.4
0.2
0.0
0.0 0.2 0.4 0.6 0.8 1.0
$\left(\frac{E}{\tau l_0^2}\right)u_x$
$x/l_0$
$u_x^{(2)} = \left(a_1 x + a_2 x^2 + a_3 x^3\right)$
$u_x^{(1)} = ax$
1.0
0.8
0.6
0.4
0.2
0.0
0.0 0.2 0.4 0.6 0.8 1.0
$\frac{\sigma}{\tau l_0}$
$x/l_0$
$u_x^{(2)} = \left(a_1 x + a_2 x^2 + a_3 x^3\right)$
$u_x^{(1)} = ax$

## II. Equation des travaux virtuels
## II.3. Principe des travaux virtuels des contraintes

**Si le champ $\underline{\underline{\sigma}}$ est en équilibre**

$$\int_{\Omega} \delta\underline{\underline{\sigma}}^{SA} \; \underline{\underline{\varepsilon}} \, dv = \int_{S_U} \left(\delta\underline{\underline{\sigma}}^{SA}\vec{n}\right)^{T} \vec{u} \, d S_{\sigma}$$

**est vérifié pour tout champ de déplacement cinématiquement admissible. Inversement si l équation précédente est statisfaite pour tout champ de contrainte statiquement admissible, l équation de compatibilité est satifaite.**

# II. Equation des travau   virtuels

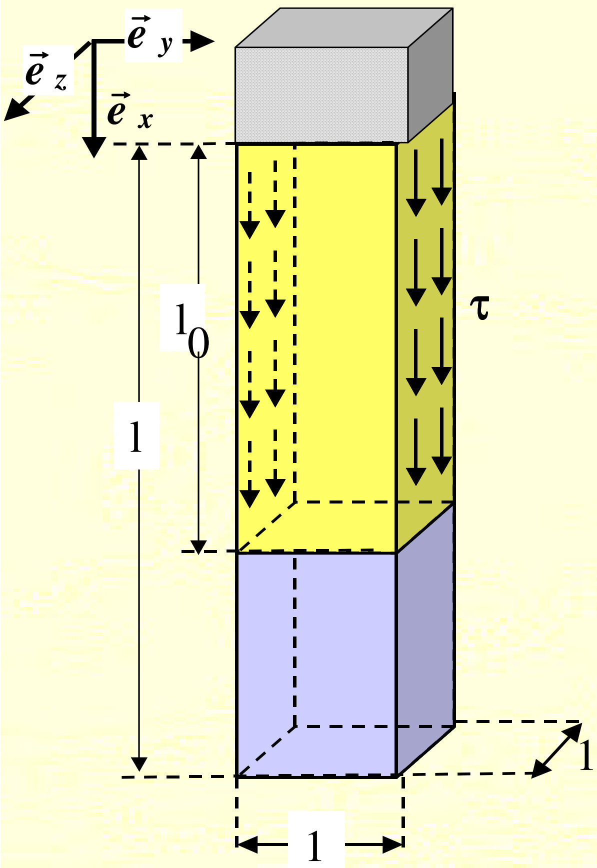
## II.3. Principe des travau   virtuels des contraintes

$$\underbrace{\begin{cases} div\left(\underline{\underline{\sigma}}\right)+\vec{f} = \quad \underline{\underline{\sigma}}^{T} = \underline{\underline{\sigma}} \ \forall \vec{x} \in \Omega \\ \underline{\underline{\sigma}}\vec{n} = \vec{t}^{\,imposé} = \overline{\vec{t}} \quad \forall \vec{x} \in S_{\sigma} \end{cases}}_{\text{satisfait e  acte   ent par hypothèse}}$$

$$\underbrace{\begin{cases} \underline{\underline{\varepsilon}} = \frac{}{2}\left[\nabla\vec{u}+\left(\nabla\vec{u}\right)^{T}\right] \forall \vec{x} \in \Omega \\ \vec{u} = \vec{u}^{\,imposé} = \overline{\vec{u}} \quad \forall \vec{x} \in S_{U} \end{cases}}_{\textit{satisfait}\ \text{appro˜i   ativve   ent}}$$

# II. Equation des travau virtuels

## II.3. Principe des travau virtuels des contraintes




$$\left\{\begin{array}{l} \sigma = - \left[\begin{array}{l} a + 2b\,y + 3cy^2 + - b + 6\tau\, y^3 \\ - \quad 6a + c\,y \end{array}\right] + b \\ \sigma_y = ay + b\,y^2 + cy^3 \\ + - b + 6\tau\, y - 6a + c\,y \end{array}\right.$$

$$\left\{\begin{array}{l} \dfrac{\partial \sigma}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} = \\ \dfrac{\partial \sigma_y}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} = \end{array}\right.$$

$$\sigma_y\, y = \frac{-}{2} = \sigma_y\, y = \frac{-}{2} = \tau$$

$$b = \frac{\tau(- + /2 - \upsilon)}{+\upsilon + 3\ 6}$$

**III. Principe du ini u de l'énergie**

**III. . Energie de défor ation et énergie co plé entaire**

**III. . . Energie de défor ation**

$$W_{él}\left(\underline{\underline{\varepsilon}}\right) = \frac{}{2}\underline{\underline{\sigma}}\left(\underline{\underline{\varepsilon}}\right)\ \underline{\underline{\varepsilon}} = \frac{}{2}\underline{\underline{\varepsilon}}\ \underline{\underline{\underline{\underline{L}}}}\ \underline{\underline{\varepsilon}} \rightarrow \underline{\underline{\sigma}} = \frac{\partial W_{él}\left(\underline{\underline{\varepsilon}}\right)}{\partial \underline{\underline{\varepsilon}}}$$

$$\delta W_{él}\left(\underline{\underline{\varepsilon}}\right) = \frac{\partial W_{él}\left(\underline{\underline{\varepsilon}}\right)}{\partial \underline{\underline{\varepsilon}}}\ \delta\underline{\underline{\varepsilon}} = \underline{\underline{\sigma}}\ \delta\underline{\underline{\varepsilon}}$$

**III. 2. Energie co plé entaire**

$$W^{c}_{él}\left(\underline{\underline{\sigma}}\right) = \frac{}{2}\underline{\underline{\sigma}}\ \underline{\underline{\underline{\underline{M}}}}\ \underline{\underline{\sigma}} \rightarrow \underline{\underline{\varepsilon}} = \frac{\partial W^{c}_{él}\left(\underline{\underline{\sigma}}\right)}{\partial \underline{\underline{\sigma}}}$$

## III. Principe du minimum de l'énergie

## III.3. Principe du minimum de l'énergie complémentaire

***Principe des travaux virtuels des contraintes***

$$\int_{\Omega} \delta \underline{\underline{\sigma}}^{SA} \quad \underline{\underline{\varepsilon}}\, dv - \int_{S_U} \left( \delta \underline{\underline{\sigma}}^{SA} \vec{n} \right)^T \vec{u}\, d S_{\sigma} =$$

***Définition de l'énergie élastique complémentaire***

$$W_{él}^{c}\left(\underline{\underline{\sigma}}\right) = \frac{1}{2} \underline{\underline{\sigma}} \quad \underline{\underline{\underline{\underline{M}}}} \quad \underline{\underline{\sigma}} \rightarrow \underline{\underline{\varepsilon}} = \frac{\partial W_{él}^{c}\left(\underline{\underline{\sigma}}\right)}{\partial \underline{\underline{\sigma}}}$$

$$\Downarrow$$

$$\underbrace{U^{c} = \int_{\Omega} W_{él}^{c}\left(\underline{\underline{\sigma}}\right) dv - \int_{S_U} \left( \underline{\underline{\sigma}} \vec{n} \right)^T \vec{u} d S}_{}$$

**δU= pour tout champ $\underline{\underline{\sigma}}^{sA}$**

## III. Principe du minimum de l'énergie

### III.3. Principe du minimum de l'énergie complémentaire

$$U = \int_{\Omega} W_{él}\left(\underline{\underline{\varepsilon}}\right) dv - \int_{S_\sigma} \vec{t}^{\,T} \vec{u}^{\,CA} d S_\sigma - \int_{\Omega} \vec{f}^{\,T} \vec{u}^{\,CA} d\Omega$$

$$U\left(\underline{\underline{\varepsilon}}^{CA}\right) \geq U\left(\underline{\underline{\varepsilon}}^{ex}\right)$$

$$U^{c} = \int_{\Omega} W^{c}_{él}\left(\underline{\underline{\sigma}}\right) dv - \int_{S_U} \left(\underline{\underline{\sigma}}\vec{n}\right)^{T} \vec{u} d S$$

$$U^{c}\left(\underline{\underline{\sigma}}^{SA}\right) \leq U^{c}\left(\underline{\underline{\sigma}}^{ex}\right)$$

## III. Principe du minimum de l’énergie
### III. . Mesure de l’erreur

$$-U\left(\underline{\underline{\varepsilon}}^{ca}\right) \leq -U\left(\underline{\underline{\varepsilon}}^{ex}\right) = U^{c}\left(\underline{\underline{\sigma}}^{ex}\right) \leq U^{c}\left(\underline{\underline{\sigma}}^{sa}\right)$$

$$U = \int_{\Omega} W_{él}\left(\underline{\underline{\varepsilon}}\right) dv - \int_{S_\sigma} \vec{\overline{t}}^{T} \vec{u}^{CA} d S_\sigma$$

$$U^{c} = \int_{\Omega} W_{él}^{c}\left(\underline{\underline{\sigma}}\right) dv - \int_{S_U} \left(\underline{\underline{\sigma}}\vec{n}\right)^{T} \vec{u} d S$$