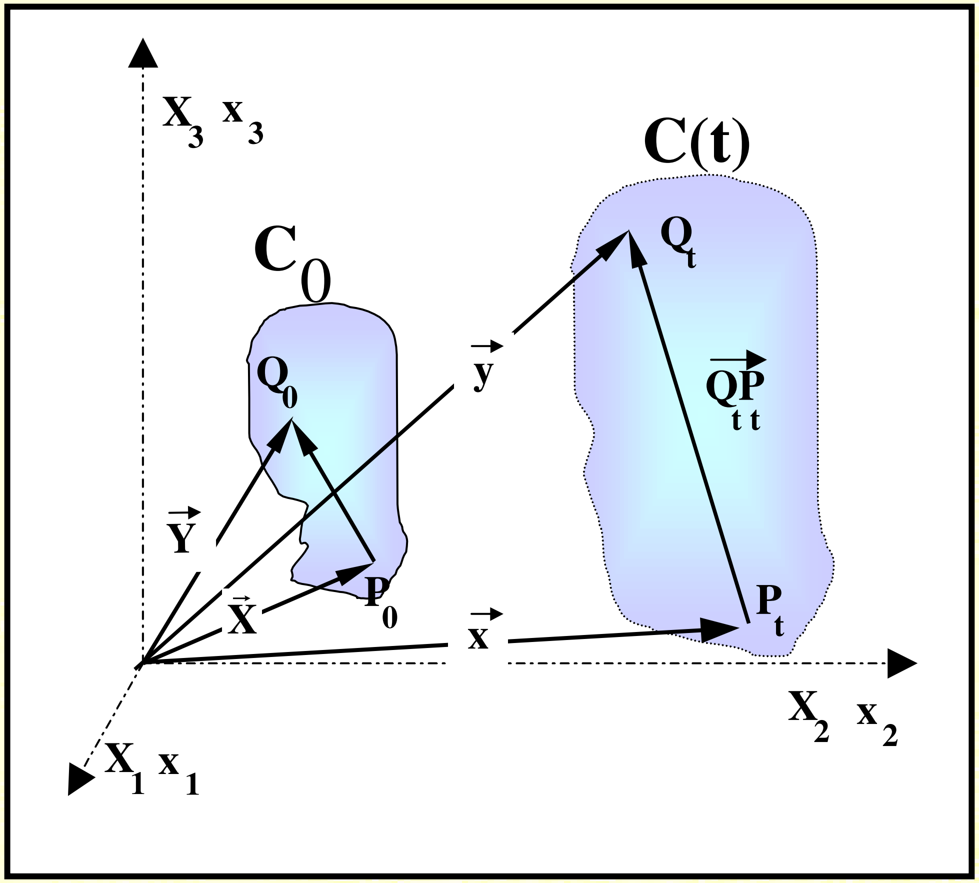
RAPPELS

I. Cinématique

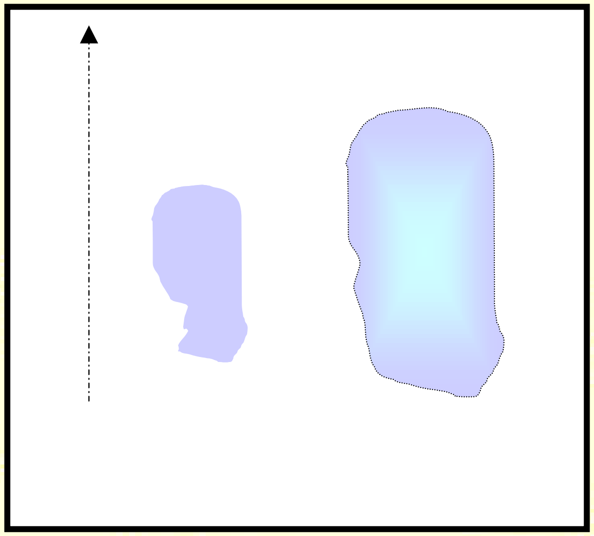


C_0 configuration initiale
description lagrangienne

$$\vec{\ } = \vec{\Phi}(\vec{X},\ )$$

C(t) configuration actuelle
description eulérienne

$$\vec{\ }\left(\vec{\ },\ \right) = \frac{\vec{\ }}{\ }$$



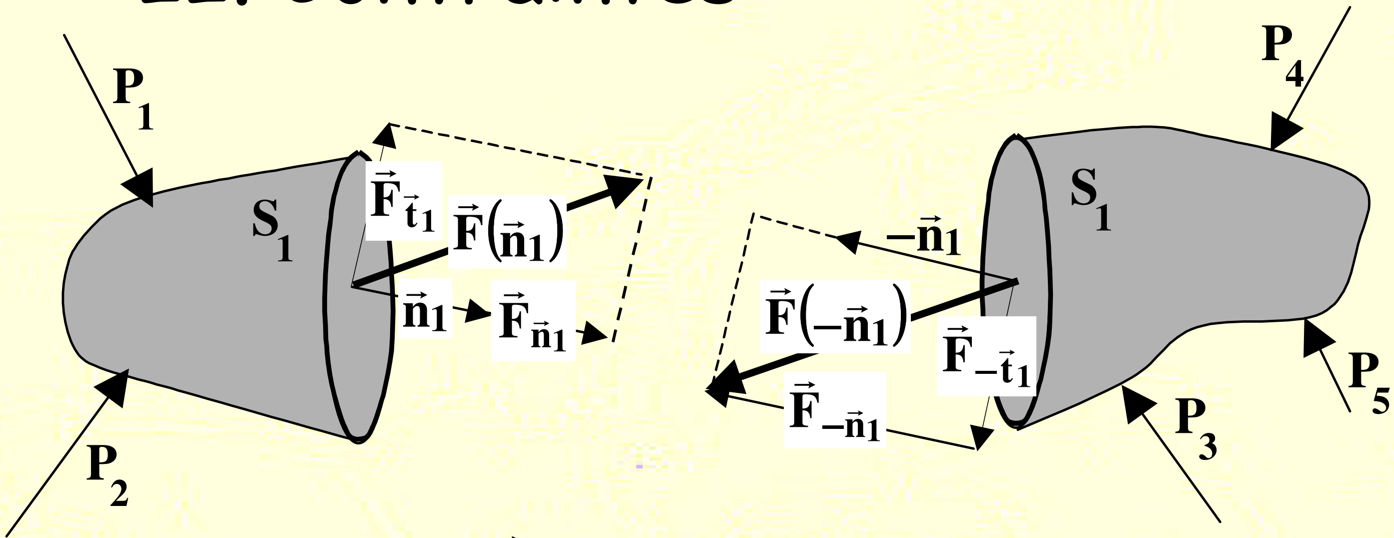
$$\underline{\underline{\;}}(\vec{X},\;)=\overrightarrow{\quad\quad}[\vec{\Phi}(\vec{X},\;)]$$

$$\rightarrow \qquad \underline{\underline{\;}} \quad \vec{X}, \quad \vec{X}$$

$$\underline{\underline{\mathbf{E}}}=\frac{1}{2}(\underline{\underline{\mathbf{F}}}^{\mathrm{T}}\underline{\underline{\mathbf{F}}}-\underline{\underline{\mathbf{I}}})$$

$$\underline{\underline{\;}}=\overrightarrow{\quad\quad\quad}\left[\vec{\Phi}\left(\vec{X},\;\right)\right]$$

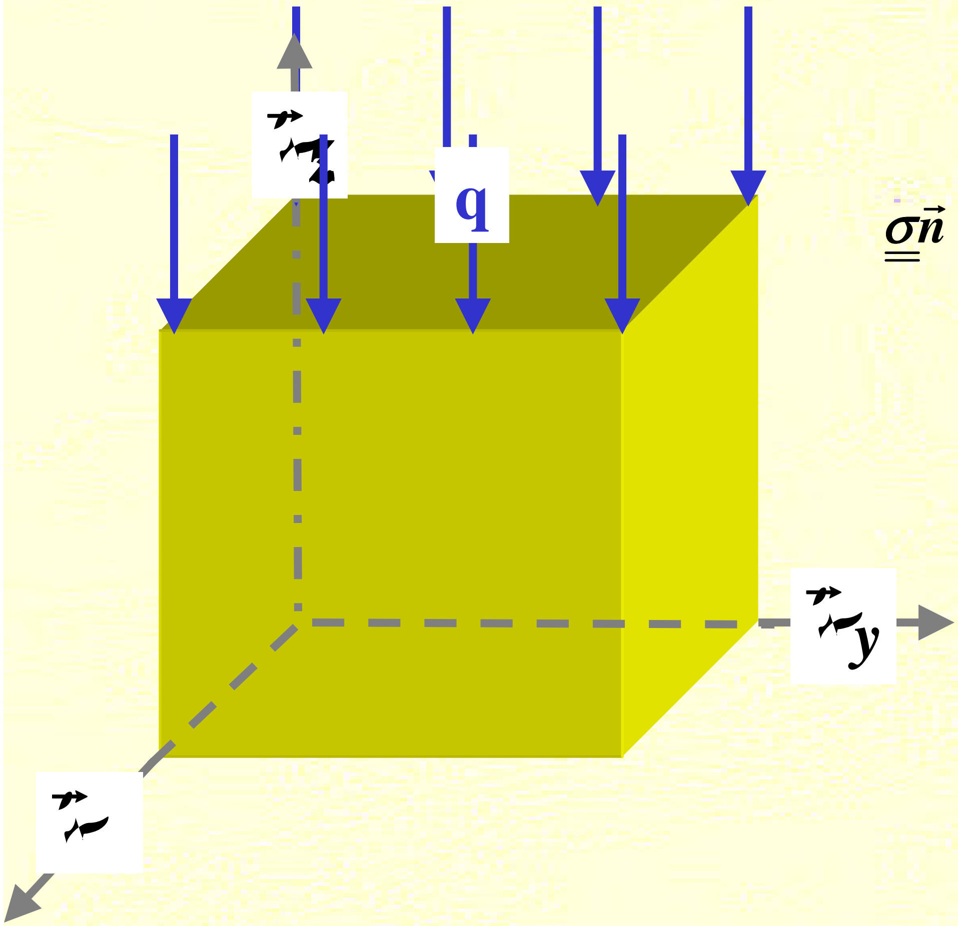
II. Contraintes



$$\vec{T} = \lim_{S \to 0} \frac{\vec{F}}{S} = T_{\vec{n}}\vec{n} + T_{\vec{t}}\vec{t}$$

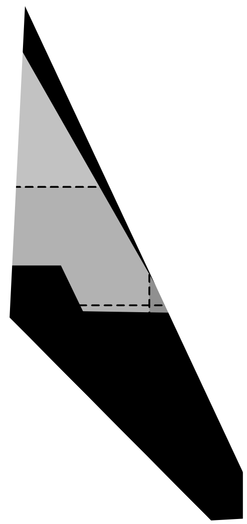
$$\vec{T}(\vec{n}) = \underline{\underline{\sigma}}\ \vec{n} \Leftrightarrow \underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \quad \vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$\vec{T}(\vec{n}) = \underline{\underline{\sigma}}\,\vec{n} = \vec{T}^{p}(\vec{n}) \text{ sur } S_\sigma$$

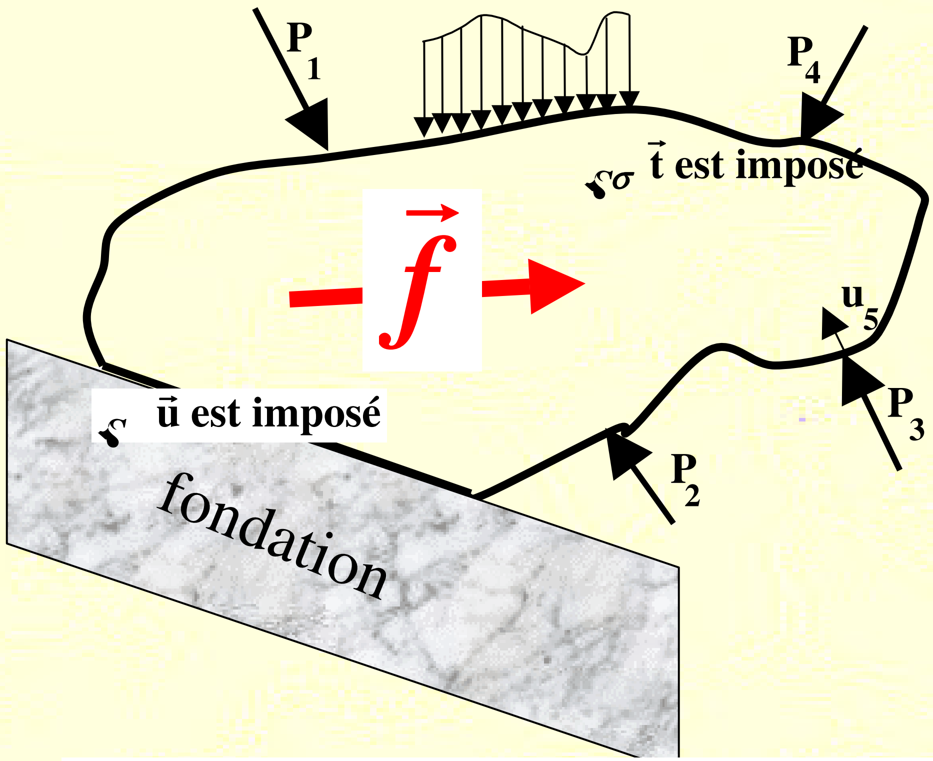


$$\underline{\underline{\sigma}}\,\vec{n} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -q \end{bmatrix}$$

$$\begin{cases} \sigma_{xz} = 0 \\ \sigma_{yz} = 0 \\ \sigma_{zz} = -q \end{cases} \quad \begin{cases} \sigma_{xx} = ? \\ \sigma_{xy} = ? \\ \sigma_{yy} = ? \end{cases}$$



III. Equations de bilan



$\vec{f} = \rho\,\vec{\ }$ par exemple

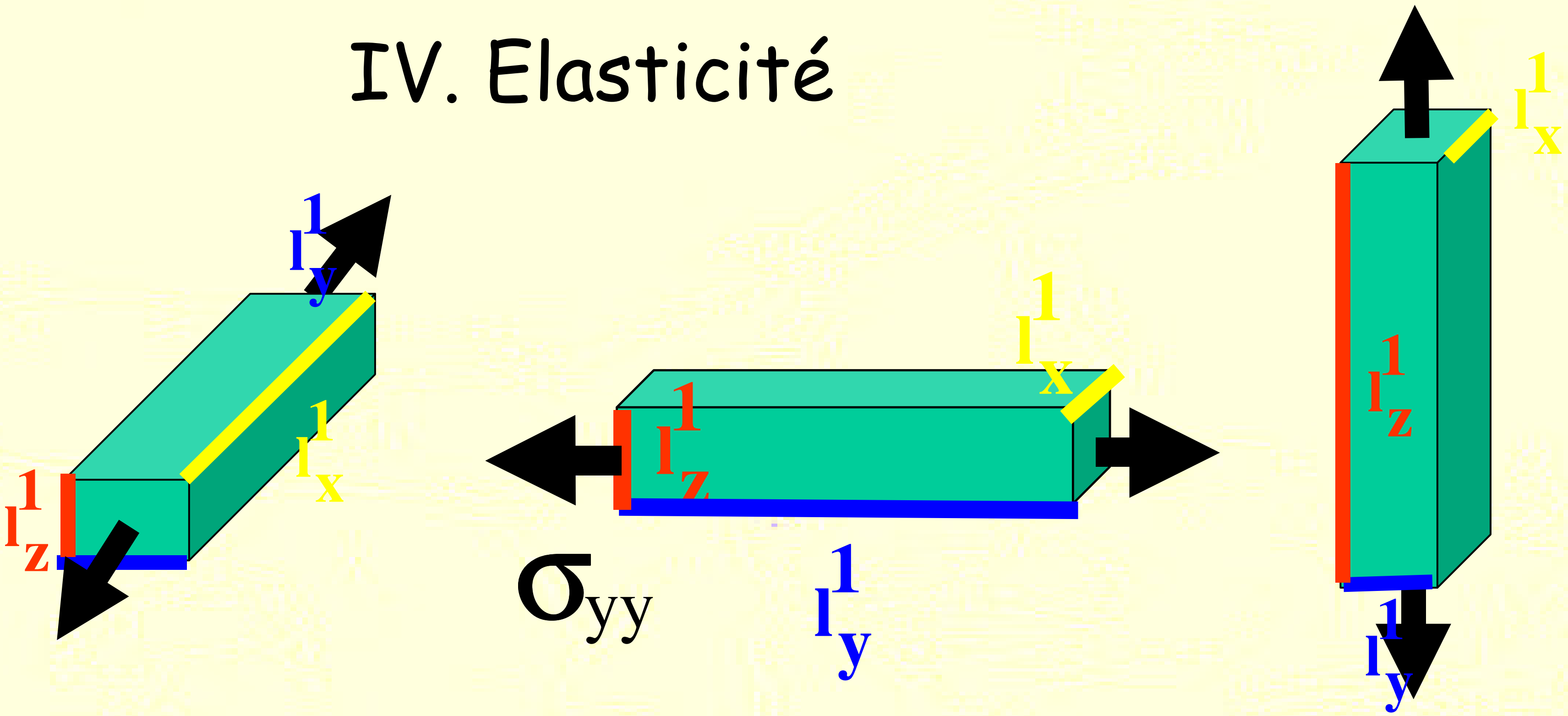
$$\int_{\Omega} \vec{f} \; + \int_{\partial\Omega_\sigma + \partial\Omega_u} \vec{\ } \; d = \int_{\Omega} \rho\vec{\gamma}$$

$$\underbrace{\int_{\Omega} \left\{ \vec{f} + \qquad \left(\underline{\underline{\sigma}}\right) - \rho\vec{\gamma} \right\} \qquad = \mathbf{0}}$$

$$\updownarrow$$

$$\vec{f} + \qquad \left(\underline{\underline{\sigma}}\right) - \rho\vec{\gamma} = \mathbf{0} \textbf{ dans } \Omega$$

IV. Elasticité



$$\varepsilon \quad = \frac{\sigma}{\quad}$$

$$\varepsilon_{yy} = -\nu\varepsilon \quad = -\nu\frac{\sigma}{\quad}$$

$$\varepsilon_{zz} = -\nu\varepsilon \quad = -\nu\frac{\sigma}{\quad}$$

$$\varepsilon \quad = -\nu\varepsilon_{yy} = -\nu\frac{\sigma_{yy}}{\quad}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{\quad}$$

$$\varepsilon_{zz} = -\nu\varepsilon_{yy} = -\nu\frac{\sigma_{yy}}{\quad}$$

$$\varepsilon \quad = -\nu\varepsilon_{zz}$$

$$\varepsilon_{yy} = -\nu\varepsilon_{zz}$$

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{\quad}$$

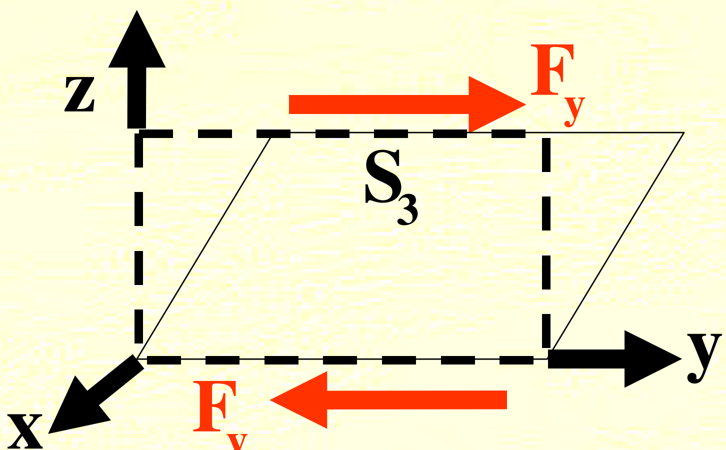
Traction le long de trois axes orthogonaux

$$\varepsilon \quad = \frac{\sigma}{\quad} - \nu \frac{\sigma_{yy}}{\quad} - \nu \frac{\sigma_{zz}}{\quad}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{\quad} - \nu \frac{\sigma}{\quad} - \nu \frac{\sigma_{zz}}{\quad}$$

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{\quad} - \nu \frac{\sigma}{\quad} - \nu \frac{\sigma_{yy}}{\quad}$$

Sollicitation en cisaillement



$$\varepsilon_{\ y} = \frac{\sigma_{\ y}}{2G}$$

$$\varepsilon_{\ z} = \frac{\sigma_{\ z}}{2G}$$

$$\varepsilon_{yz} = \frac{\sigma_{yz}}{2G}$$

$$G = \frac{}{2(1+\nu)}$$

$$
\begin{bmatrix} \begin{pmatrix} \varepsilon \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{pmatrix} \\ \begin{pmatrix} \varepsilon_{\ z} \\ \varepsilon_{yz} \\ \varepsilon_{\ y} \end{pmatrix} \end{bmatrix}
= \frac{1}{\ }
\begin{bmatrix}
\begin{pmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{pmatrix} & \mathbf{0} \\
\mathbf{0} & \begin{pmatrix} 1+\upsilon & 0 & 0 \\ 0 & 1+\upsilon & 0 \\ 0 & 0 & 1+\upsilon \end{pmatrix}
\end{bmatrix}
*
\begin{bmatrix} \begin{pmatrix} \sigma \\ \sigma_{yy} \\ \sigma_{zz} \end{pmatrix} \\ \begin{pmatrix} \sigma_{\ z} \\ \sigma_{yz} \\ \sigma_{\ y} \end{pmatrix} \end{bmatrix}
$$

La loi de Hooke complète pour un matériau anisotrope

$$\left.\begin{aligned} \sigma &= L \quad \varepsilon \\ \sigma &= \sigma \\ \varepsilon &= \varepsilon \end{aligned}\right\} \begin{aligned} L &= L \\ L &= L \end{aligned}$$

IV. Energie de déformation

$$= \int_{\blacktriangle} \sigma \; \Delta\varepsilon \qquad\qquad = \int_{\blacktriangle} \int_{0}^{\varepsilon} \sigma \quad \varepsilon$$

$$= \int_{0}^{\varepsilon} \sigma \quad \varepsilon$$

$$= \int_{0}^{\varepsilon} \vec{\sigma} \, \blacktriangle \; \vec{\varepsilon}$$

IV. L’énergie de déformation

IV.1. Expression de l’énergie en fonction des contraintes et des déformations

$$= \int_0^{\varepsilon} \sigma \quad \varepsilon$$

$$\varepsilon = M \quad \sigma$$

$$\sigma = L \quad \varepsilon$$

$$\rightarrow$$

$$= \frac{1}{2} \underline{\underline{\sigma}} : \underline{\underline{\underline{\underline{M}}}} : \underline{\underline{\sigma}}$$

$$= \frac{1}{2} \underline{\underline{\varepsilon}} : \underline{\underline{\underline{\underline{L}}}} : \underline{\underline{\sigma}}$$

IV. L’énergie de déformation

IV.1. Propriétés de M et L

$$= \frac{1}{2}\vec{\sigma}^{T} M \vec{\sigma} \geq 0 \rightarrow M \quad f\,n\;\;p\quad f$$

$$= \frac{1}{2}\vec{\varepsilon}^{T} L \vec{\varepsilon} \geq 0 \rightarrow L \quad f\,n\;\;p\quad f$$

Chapitre V : Méthodes semi-inverses

I. Bilan

I.1. Nombre d’inconnus

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Vecteur déplacement $\vec{\ } = \begin{pmatrix} \\ y \\ z \end{pmatrix}$ $\Rightarrow$ **3**

I. Bilan

I.1. Nombre d'inconnus

Vecteur déplacement $\vec{\ } = \begin{pmatrix} \\ y \\ z \end{pmatrix}$ → **3**

Tenseur petites déformations $\vec{\varepsilon} = \begin{bmatrix} \varepsilon \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{\ z} \\ \varepsilon_{yz} \\ \varepsilon_{\ y} \end{bmatrix}$ $\underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \sigma_{22} & \sigma_{23} \\ \varepsilon_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}$ → **6**

I. Bilan

I.1. Nombre d'inconnus

Vecteur déplacement $\vec{\ } = \begin{pmatrix} \\ _{y} \\ _{z} \end{pmatrix}$ $\longrightarrow$ **3**

Tenseur petites déformations $\vec{\varepsilon} = \begin{bmatrix} \varepsilon \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{\ z} \\ \varepsilon_{yz} \\ \varepsilon_{\ y} \end{bmatrix}$ $\underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \sigma_{22} & \sigma_{23} \\ \varepsilon_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}$ $\longrightarrow$ **6**

Tenseur contraintes de Cauchy $\vec{\sigma} = \begin{bmatrix} \sigma \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{\ z} \\ \sigma_{yz} \\ \sigma_{\ y} \end{bmatrix}$ $\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$ $\longrightarrow$ **6**

I. Bilan

I.2. Nombre d'équations en volume

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q q n c

$$\partial\sigma / \partial + f = 0$$

I. Bilan

I.2. Nombre d'équations en volume

q , – *q* *n* *c*

$$\partial\sigma \;/\partial \quad + f \;= 0$$

p , – *q* *n* *c*

$$\frac{\partial^2}{\partial \quad \partial}\varepsilon + \frac{\partial^2}{\partial \quad \partial}\varepsilon - \frac{\partial}{\partial}\left(\frac{\partial}{\partial}\varepsilon \quad\right) - \frac{\partial^2}{\partial \quad \partial}\varepsilon \quad = 0$$

I. Bilan

I.2. Nombre d'équations en volume

q - *q* *n* *c*

$$\partial\sigma \;\; / \partial \;\;\; + f \;\; = 0$$

p - *q* *n* *c*

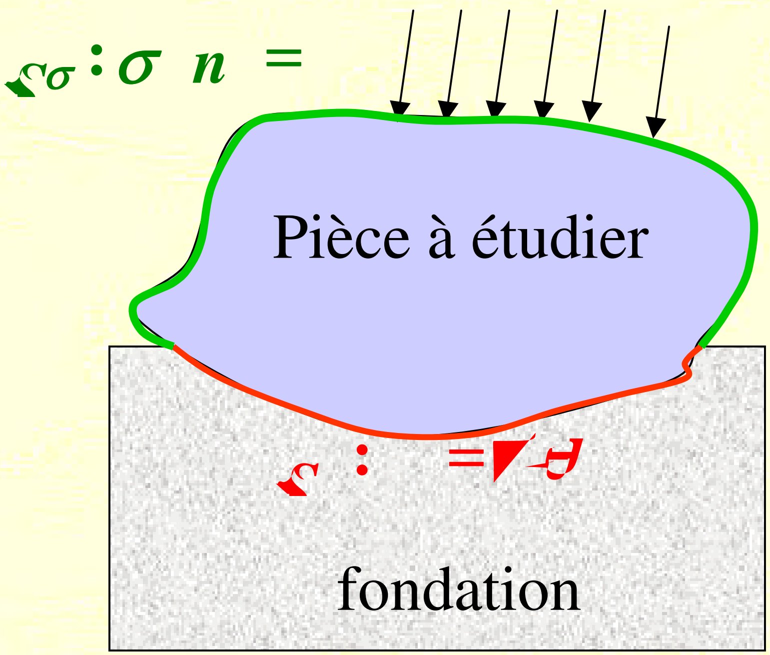
$$\frac{\partial^2}{\partial \;\; \partial}\varepsilon \; + \frac{\partial^2}{\partial \;\; \partial}\varepsilon \;\; - \frac{\partial}{\partial}\left(\frac{\partial}{\partial}\varepsilon \;\; \right) - \frac{\partial^2}{\partial \;\; \partial}\varepsilon \;\; = 0$$

L - *q* *n* *c*

$$\varepsilon \;\; = \frac{1}{\;}\left\{(1+\nu)\sigma \;\; - \upsilon\sigma \;\; \delta \;\; \right\}$$

I. Bilan

I.2. Nombre de conditions limites



q *n* *c*

σ *n* =

II. Equations fondamentales de l'élasticité

II.1. Démarche

15 équations différentielles du premier ordre

équilibre : $\partial\sigma\ /\partial\ \ + f\ = 0$

compatibilité : $\varepsilon\ = \frac{1}{2}\left(\frac{\partial}{\partial} + \frac{\partial}{\partial}\right)$

Hooke : $\varepsilon\ = \frac{1}{\ }\left\{(1+\nu)\sigma\ - \upsilon\sigma\ \delta\ \right\}$

3 équations différentielles scalaires du second ordre

sur $\underline{\underline{\sigma}}$ (équations de Beltrami-Mitchel)

ou

sur $\vec{u}$ (équations de Navier)

II. Equations fondamentales de l'élasticité

II.2. Equation de Beltrami-Mitchell

II.2.1. Cas général

$$\Delta\varepsilon + \overrightarrow{\quad}\left\{\overrightarrow{\quad}\left[\quad c'(\underline{\underline{\varepsilon}})\right]\right\}$$

$$- \overrightarrow{\quad}\left[\quad (\underline{\underline{\varepsilon}})\right] - \overrightarrow{\quad}\left[\quad (\underline{\underline{\varepsilon}})\right]^T = 0$$

$$\underline{\underline{\varepsilon}} = \frac{1}{\quad}\left\{(1+\nu)\underline{\underline{\sigma}} - \upsilon \quad c'(\underline{\underline{\sigma}})\,\underline{\underline{\ }}\right\}$$

$$\overrightarrow{\quad}(\underline{\underline{\sigma}}) + \vec{f} = 0$$

$$\rightarrow$$

$$(1+\nu)\Delta\underline{\underline{\sigma}}$$

$$+ \overrightarrow{\quad}\left\{\overrightarrow{\quad}\left[\quad c'(\underline{\underline{\sigma}})\right]\right\}$$

$$+ (1+\nu)\left\{ \begin{array}{l} \overrightarrow{\quad}\left[\vec{f}\right] \\ + \left\{\overrightarrow{\quad}\left[\vec{f}\right]\right\}^T \end{array} \right\}$$

$$+ \quad \left[\vec{f}\right]\underline{\underline{\ }} = 0$$

II. Equations fondamentales de l'élasticité

II.2. Equation de Beltrami-Mitchell

II.2.1. Cas général

sec n $\underline{\underline{\sigma}}$

$$(1+\nu)\Delta\underline{\underline{\sigma}} + \longrightarrow \left\{ \longrightarrow \left[\overbrace{c\left(\underline{\underline{\sigma}}\right)}^{\text{role particulier}} \right] \right\}$$

$$+(1+\nu)\left\{ \longrightarrow \left[\vec{f}\right] + \left\{ \longrightarrow \left[\vec{f}\right] \right\}^{T} \right\} + \left[\vec{f}\right]_{=} = 0$$

p $\vec{f}$

II. Equations fondamentales de l'élasticité

II.2. Equation de Beltrami-Mitchell

II.2.2. Forces de volume nulles ou constantes

$$(1+\nu)\Delta\underline{\underline{\sigma}} + \overrightarrow{} \left\{ \overrightarrow{} \quad c' \left(\underline{\underline{\sigma}}\right)\right\} = 0$$

$$\Delta\sigma \quad = \Delta\left[trace\left(\underline{\underline{\sigma}}\right)\right] 0$$

$$\Delta\Delta\sigma \quad = \Delta\Delta\underline{\underline{\sigma}} = 0$$

II. Equations fondamentales de l’élasticité

II.2. Equation de Beltrami-Mitchell

II.2.3. Application : contraintes planes ou déformations planes

$$\left(\underline{\underline{\sigma}}\right) = 0 \rightarrow \begin{cases} \sigma \quad = \dfrac{\partial \varphi^2}{\partial y^2} \\ \sigma_{\ y} = -\dfrac{\partial \varphi^2}{\partial \ \partial y} \\ \sigma_{yy} = \dfrac{\partial \varphi^2}{\partial \ ^2} \end{cases}$$

[redacted]

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left\{ \qquad \left(\vec{\ }\right) + \left[\qquad \left(\vec{\ }\right)\right]^{\cdot} \right\}$$

$$\underline{\underline{\sigma}} = 2G\left\{ \underline{\underline{\varepsilon}} + \frac{\upsilon}{1-2\upsilon} \quad c\left(\underline{\underline{\varepsilon}}\right)\underline{\underline{\ }} \right\}$$

II. Equations fondamentales de l'élasticité

II.3. Equation de Navier

II.3.1. Cas général

$$\left.\begin{array}{l}
(\underline{\underline{\sigma}}) = \vec{f} \\[2em]
\underline{\underline{\varepsilon}} = \dfrac{1}{2}\left\{ \begin{array}{l} (\vec{\ }) \\ +[\quad (\vec{\ })]^{T} \end{array} \right\} \\[2em]
\left\{ \begin{array}{l} \underline{\underline{\sigma}} = 2G\left\{ \underline{\underline{\varepsilon}} + \dfrac{\nu}{1-2\nu}\theta \,\underline{\underline{\ }} \right\} \\ \theta = c(\underline{\underline{\varepsilon}}) \end{array} \right.
\end{array} \right\} \rightarrow$$

$$\overbrace{\Delta \vec{\ } + \underbrace{\left(\frac{1}{1-2\nu}\right)}_{c \quad p \quad - \quad n} \longrightarrow \left[\quad (\vec{\ })\right]}^{p\,n \quad c \quad p \quad - \quad p \quad c - \quad n}$$

$$+ \frac{\overbrace{\vec{f}}^{f \quad c - \quad q \quad -}}{\underbrace{G}_{c \quad p \quad - \quad n}} = 0$$

II. Equations fondamentales de l'élasticité

II.3. Equation de Navier

II.3.1. Cas particuliers

a) Forces de volume nulles

$$\Delta \quad + \underbrace{\left(\frac{1}{1-2\nu}\right)}_{c \quad p \quad n} \quad \left[\quad (\ ) \right] = 0$$

II. Equations fondamentales de l'élasticité

II.3. Equation de Navier

II.3.1. Cas particuliers

a) Forces de volume dérivant d'un potentiel

$$\Delta \vec{\ } + \left(\frac{1}{1-2\nu}\right) \overrightarrow{\qquad} \left[\quad \left(\vec{\ } \right) \right] + \frac{\vec{f}}{G} = 0$$

$$\vec{f} = \overrightarrow{\qquad} \left(\quad \right)$$

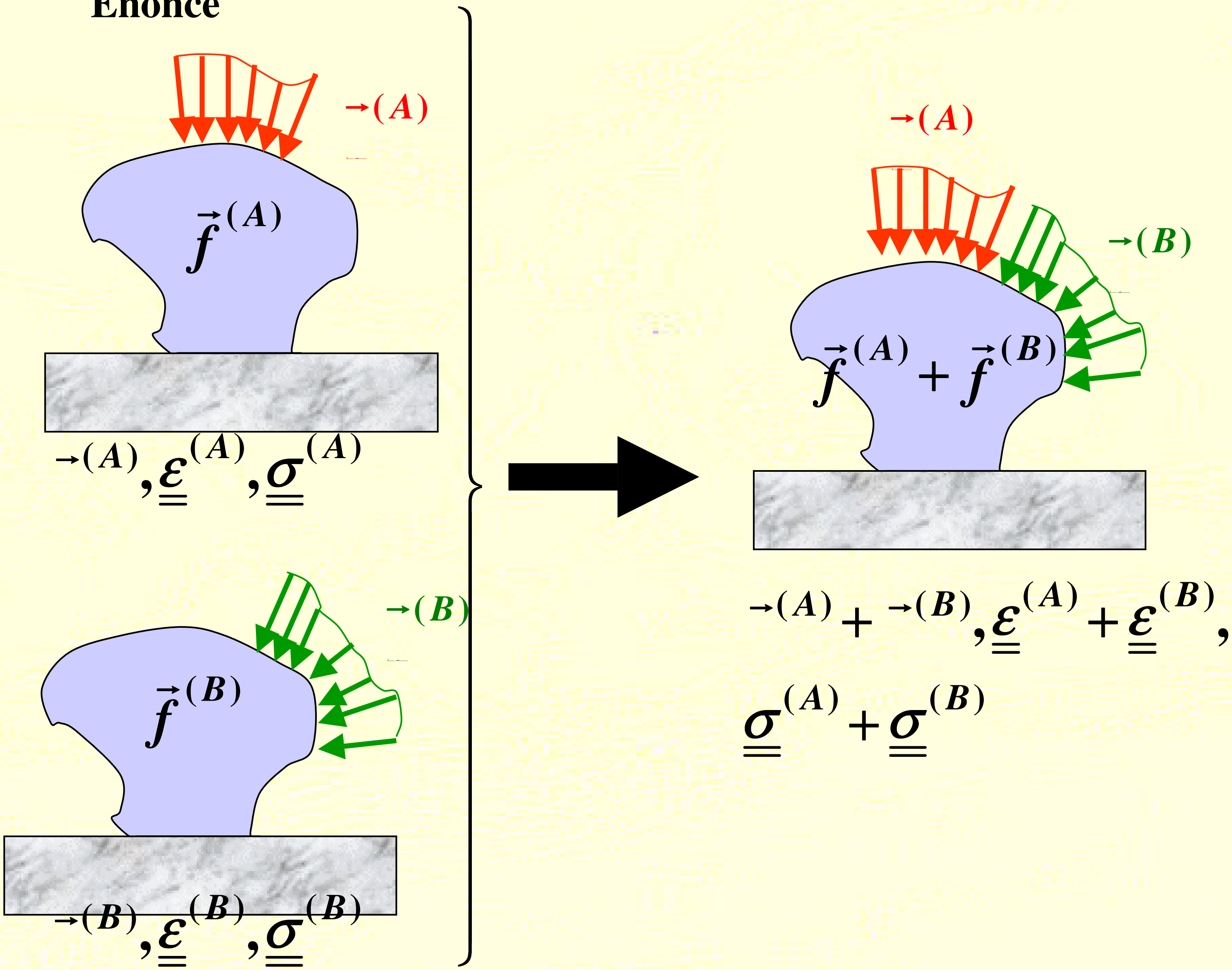
$$\Delta \vec{\ } + \frac{1}{1-2\nu} \overrightarrow{\qquad} \left[\quad \left(\vec{\ } \right) \right]$$

$$+ \frac{\overrightarrow{\qquad} \left(\quad \right)}{G} = 0$$

III. Principe de superposition et unicité de la solution

III.1. Principe de superposition

Enoncé



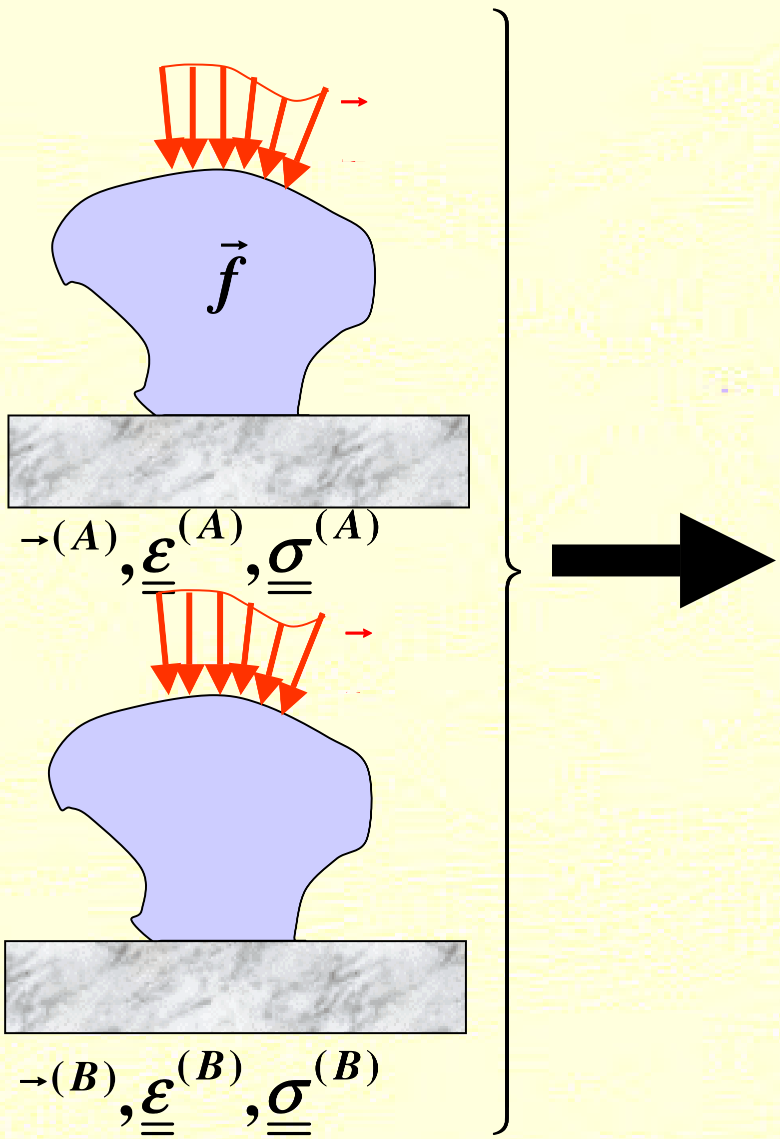
III. Principe de superposition et unicité de la solution

III.1. Principe de superposition

limitation : hypothèse des petites perturbations

III. Principe de superposition et unicité de la solution

III.2. Unicité de la solution



$$\vec{\ }^{(A)} - \vec{\ }^{(B)} = 0?$$

$$\underline{\underline{\varepsilon}}^{(A)} - \underline{\underline{\varepsilon}}^{(B)} = 0?$$

$$\underline{\underline{\sigma}}^{(A)} - \underline{\underline{\sigma}}^{(B)} = 0?$$

III. Principe de superposition et unicité de la solution

III.2. Unicité de la solution

$$\left.\begin{array}{l} \vec{\ }' = \vec{\ }^{(A)} - \vec{\ }^{(B)} \\ \underline{\underline{\varepsilon}}' = \underline{\underline{\varepsilon}}^{(A)} - \underline{\underline{\varepsilon}}^{(B)} \\ \underline{\underline{\sigma}}' = \underline{\underline{\sigma}}^{(A)} - \underline{\underline{\sigma}}^{(B)} \end{array}\right\}$$

est solution du problème

$$\left\{\begin{array}{l} \vec{T}' = \vec{T} - \vec{T} = 0 \\ \vec{f}' = \vec{f} - \vec{f} = 0 \end{array}\right.$$

$$\int \vec{\sigma}'^{T} \underline{\underline{M}} \vec{\sigma}' = \int \vec{T}'^{T} \vec{\ }' \, S = 0$$

$$\left.\begin{array}{l} \displaystyle\int \underbrace{\left[\vec{\sigma}^{A} - \vec{\sigma}^{B}\right]^{T}}_{\vec{\sigma}'} \underline{\underline{M}} \underbrace{\left[\vec{\sigma}^{A} - \vec{\sigma}^{B}\right]}_{\vec{\sigma}'} = 0 \\ \underline{\underline{M}} \quad f \, n \quad p \quad f \end{array}\right\} \rightarrow \vec{\sigma}^{A} - \vec{\sigma}^{B} = 0$$